Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 6 Verifications

1. Harmonic functions. Verify theorem 1, p. 460, for $f = 2z^2 - x^2 - y^2$ and S the surface of the box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Theorem 1, p. 460, is sort of squeezed in by itself. The function f in the problem description is a harmonic function whose domain is enclosed in a space D, which here I can consider to be ℝ. S is a "piecewise smooth closed orientable surface in D", here a cuboid which intersects both leaves of the problem function. Then the assertion, to be verified, is that the integral of the normal derivative of f taken over S is equal to zero.

```
Clear["Global`*"]
```

First I will plot the surroundings.

```
innen = 2 z^2 - x^2 - y^2
```

$$-x^2 - y^2 + 2z^2$$

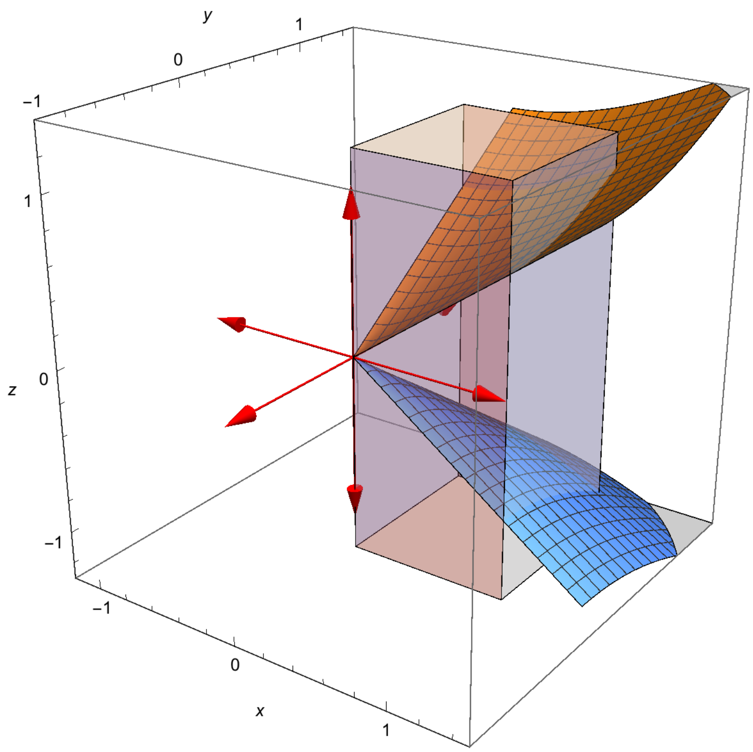
```
inne = Solve[2 z^2 - x^2 - y^2 == 0, z]
```

$$\left\{\left\{z \to -\frac{\sqrt{x^2+y^2}}{\sqrt{2}}\right\}, \left\{z \to \frac{\sqrt{x^2+y^2}}{\sqrt{2}}\right\}\right\}$$

```
fir = Plot3D[{Sqrt[x^2 + y^2]/Sqrt[2], -Sqrt[x^2 + y^2]/Sqrt[2]}, {x, 0, 1.5}, {y, 0, 1.5},
   PlotRange → {{-1.2, 1.5}, {-1, 1.5}, {-1.3, 1.4}}];

sec = Graphics3D[{{Opacity[.4], Cuboid[{0, 0, -1.2}, {1, 1, 1.2}]},
    {Red, Arrowheads[0.04], Arrow[Tube[{{0, 0, 0}, {0, 0, 1}}, 0.006]]},
    {Red, Arrowheads[0.04], Arrow[Tube[{{0, 0, 0}, {1, 0, 0}}, 0.006]]},
    {Red, Arrowheads[0.04], Arrow[Tube[{{0, 0, 0}, {0, 1, 0}}, 0.006]]},
    {Red, Arrowheads[0.04], Arrow[Tube[{{0, 0, 0}, {0, -1, 0}}, 0.006]]},
    {Red, Arrowheads[0.04], Arrow[Tube[{{0, 0, 0}, {-1, 0, 0}}, 0.006]]},
    {Red, Arrowheads[0.04],
     Arrow[Tube[{{0, 0, 0}, {0, 0, -1}}, 0.006]]}}];
```

```
Show[ fir, sec, BoxRatios → {1, 1, 1}, AxesLabel -> {x, y, z}]
```



Okay, the cuboid is the “closed orientable surface”, consisting of, in fact, six surfaces, and a total of six normal vectors. The surface closest to the viewer is s4, and its normal vector, {0,-1,0}, has the closest arrowhead.

In the general case the cuboid has sides of length a, b, and c, but the normal vectors can be the expected ones, normalized. However, in the first run through, I’m going to examine a cuboid with unit length faces, because it comes out neatly that way.

```
s1 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{0, 0, 1}
```

$4z$

```
s2 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{1, 0, 0}
```

$-2x$

```
s3 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{0, 1, 0}
```

$-2y$

```
s4 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{0, -1, 0}
```

$2y$

```
s5 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{-1, 0, 0}
2 x

s6 = Grad[2 z^2 - x^2 - y^2, {x, y, z}].{0, 0, -1}
-4 z

tot = s1 + s2 + s3 + s4 + s5 + s6
0
```

The above answer agrees with the text's, without any integrating. However, I should look at the more general case, with general cuboid side lengths.

```
g1 = {"surface", "dot result",
   "w/ substitute", "surface int", "surf.int. result"};

g2 = {{"s1", s1, s1 /. z -> c, "∫_0^b ∫_0^a 4cdxdy", ∫_0^b ∫_0^a 4 c dx dy},
   {"s2", s2, s2 /. x -> a, "∫_0^c ∫_0^b -2adydz", ∫_0^c ∫_0^b -2 a dy dz},
   {"s3", s3, s3 /. y -> b, "∫_0^c ∫_0^a -2bdxdz", ∫_0^c ∫_0^a -2 b dx dz},
   {"s4", s4, s3 /. y -> 0, "∫∫_{s4} ∂f/∂dn dA", 0}, {"s5", s5, s5 /. x -> 0,
    "∫∫_{s5} ∂f/∂dn dA", 0}, {"s6", s6, s6 /. z -> 0, "∫∫_{s6} ∂f/∂dn dA", 0}};
```

What the grid below shows is the same goose egg result in the general case. When performing the substitution representing the general cuboid side length, it is necessary to do the surface integrals. These turn out as hoped, validating the method. Most of this was borrowed from the s.m., except that surface designations and dotting partners differed. (The symbol "dn" represents the relevant dotted normal expression.)

```
Grid[Prepend[g2, g1], Frame -> All]
```

surface	dot result	w/ substitute	surface int	surf.int. result
s1	4 z	4 c	$\int_0^b\int_0^a 4c\,dx\,dy$	4 a b c
s2	-2 x	-2 a	$\int_0^c\int_0^b -2a\,dy\,dz$	-2 a b c
s3	-2 y	-2 b	$\int_0^c\int_0^a -2b\,dx\,dz$	-2 a b c
s4	2 y	0	$\int\int_{s_4}\frac{\partial f}{\partial dn}dA$	0
s5	2 x	0	$\int\int_{s_5}\frac{\partial f}{\partial dn}dA$	0
s6	-4 z	0	$\int\int_{s_6}\frac{\partial f}{\partial dn}dA$	0

3. Green's first identity. Verify numbered line (8), p. 461, for f = $4y^2$, g = x^2, S the surface of the "unit cube", 0 ≤ x ≤ 1, 0 ≤ y ≤ 1, 0 ≤ z ≤ 1. What are the assumptions on f and g in numbered line (8)? Must f and g be harmonic?

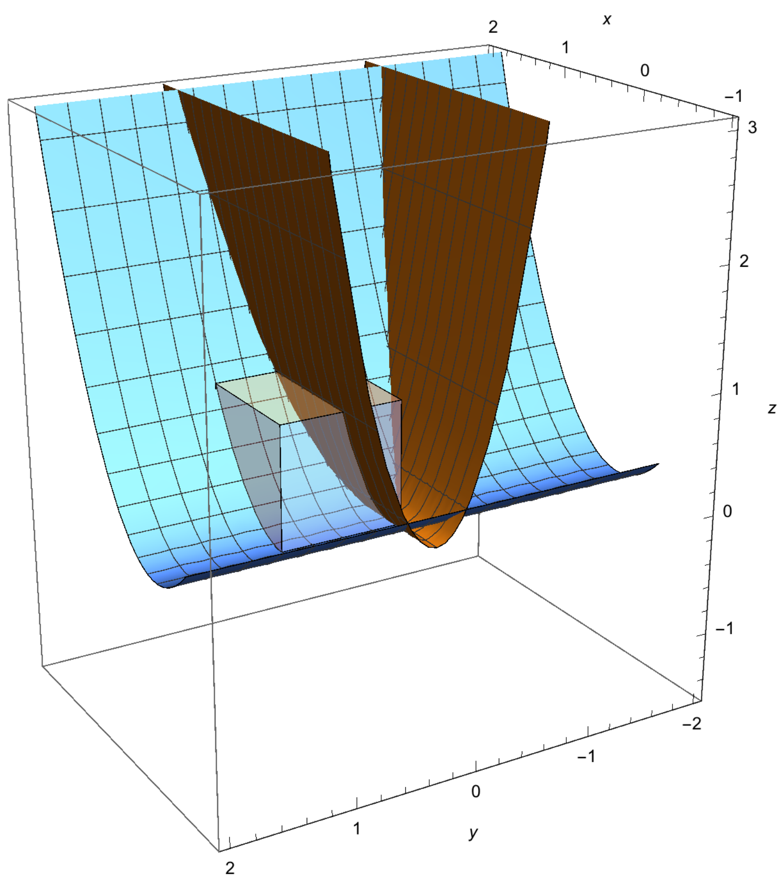
```
Clear["Global`*"]
```

I should put down Green's first identity for consideration

$$\int\int_T\int \left(f\, \nabla^2 g + \mathrm{Grad}[f].\mathrm{Grad}[g]\right)\, dV = \int_S\int f\, \frac{\partial g}{\partial n}\, dA$$

The nabla squared is the Laplacian, and $\frac{\partial g}{\partial n}$ is the function dotted with the normal vector. The use of this equation is in situations where a boundary might be easier to find than a volume, or vice versa. It's good to have the option. First I will try to sketch out the environment as described in the problem.

```
p1 = Plot3D[{4 y^2, x^2}, {x, -0.5, 2}, {y, -2, 2}];
p2 = Graphics3D[{{Opacity[.4], Cuboid[{0, 0, 0}, {1, 1, 1}]}}];
Show[ p1, p2, BoxRatios → {0.75, 1, 1.1}, AxesLabel -> {x, y, z},
 PlotRange → {{-1, 2}, {-2, 2}, {-1.5, 3}}, ImageSize → 400]
```



First trying to handle the volume side of the identity, the one with three integral trees.

```
f[x_, y_, z_] = 4 y^2
4 y^2
```

```
g[x_, y_, z_] = x^2
x^2
```

$$\int_0^1\int_0^1\int_0^1 (\mathrm{f}[x, y, z]\ \mathrm{Laplacian}[\mathrm{g}[x, y, z], \{x, y, z\}] + \mathrm{Grad}[\mathrm{f}[x, y, z], \{x, y, z\}].\mathrm{Grad}[\mathrm{g}[x, y, z], \{x, y, z\}])\, dx\, dy\, dz$$

$$\frac{8}{3}$$

The answer above matches that in the text. Now to see if I can get the same thing on the rhs. The cuboid has six surface faces, and I need to check $\frac{\partial g}{\partial n}$ for each one. They are all zero except for s3 and s4, but when I substitute in the x values there, s4 disappears, because for the surface corresponding to that normal, x = 0.

```
s1 = Grad[x^2, {x, y, z}].{0, 0, 1}
0
```

```
s2 = Grad[x^2, {x, y, z}].{0, 1, 0}
0
```

```
s3 = Grad[x^2, {x, y, z}].{1, 0, 0}
2 x
```

```
s4 = Grad[x^2, {x, y, z}].{-1, 0, 0}
-2 x
```

```
s5 = Grad[x^2, {x, y, z}].{0, -1, 0}
0
```

```
s6 = Grad[x^2, {x, y, z}].{0, 0, -1}
0
```

Now I am left to calculate the double integral with the remaining constituents.

$$\int_0^1\int_0^1 (\mathrm{f}[x, y, z])\, 2\, dx\, dy$$

$$\frac{8}{3}$$

The above answer matches that in the text. As for the question about the conditions of the described functions, the requirements for the environment and functions are for a bounded solid region with a piecewise continuous boundary surface. The numbered line intro says that f grad g satisfies the divergence theorem, which requires harmonic functions. So I believe they would need to be harmonic.

5. Green's second identity. Verify numbered line (9), p. 461, for $f = 6y^2$, $g = 2x^2$, S the unit cube in problem 3.

```
Clear["Global`*"]
```

I could start by putting down numbered line (9).

$$\iiint_T \left(f\,\nabla^2 g - g\,\nabla^2 f\right) dV = \iint_S \left(f\,\frac{\partial g}{\partial n} - g\,\frac{\partial f}{\partial n}\right) dA$$

The given functions are not exactly the same as the last problem, but very close. They are close enough that it does not make much sense in redrawing the scene.

```
f[x_, y_, z_] = 6 y^2
6 y^2

g[x_, y_, z_] = 2 x^2
2 x^2
```

$$\int_0^1\int_0^1\int_0^1 (\texttt{f[x, y, z] Laplacian[g[x, y, z], \{x, y, z\}]} - \texttt{g[x, y, z] Laplacian[f[x, y, z], \{x, y, z\}]})\, dx\, dy\, dz$$

```
0
```

The above cell matches the answer in the text.

I have to make two complete lists of dot products for the cuboid, and I'm wondering if I could save space using tables.

```
cub = {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}, {-1, 0, 0}, {0, -1, 0}, {0, 0, -1}};

rsurf = {s1, s2, s3, s4, s5, s6};
```

The following table is for g, and the only non-zero member is s3.

```
t1 = Table[{rsurf[[n]], Grad[2 x^2, {x, y, z}].cub[[n]]}, {n, 1, 6}]
{{s1, 0}, {s2, 0}, {s3, 4 x}, {s4, -4 x}, {s5, 0}, {s6, 0}}
```

The following table is for f, and the only non - zero member is s2.

```
t2 = Table[{rsurf[[n]], Grad[6 y^2, {x, y, z}].cub[[n]]}, {n, 1, 6}]
{{s1, 0}, {s2, 12 y}, {s3, 0}, {s4, 0}, {s5, -12 y}, {s6, 0}}
```

Now to plug in to the rhs of the identity.

$$\int_0^1\int_0^1 ((f[x, y, z])\, 4 - (g[x, y, z])\, 12)\, dx\, dy$$

0

The green cell above matches the answer in the text, and verifies identity 2 for the given functions.

7 - 11 Volume
Use the divergence theorem, assuming that the assumptions on T and S are satisfied.

7. Show that a region T with boundary surface S has the volume

$$V = \int_S\int x\, dy\, dz = \int_S\int y\, dz\, dx = \int_S\int z\, dx\, dy = \frac{1}{3}\int_S\int (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

The way the above equations are constituted, it is really only necessary to demonstrate the first three out of four quantities, but I'm not capable of doing so right now.

For the rest I am going to rely on *https://www.math24.net/divergence-theorem/* for the descriptive details. I'm going to:
Let G be a three-dimensional solid bounded by a piecewise smooth closed surface S that has orientation pointing out of G and let

F (x, y, z) = (P (x, y, z), Q (x, y, z), R (x, y, z))

be a vector field whose components have continuous partial derivatives.

$$\nabla .F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of the vector field F (it's also denoted div F) and the surface integral is taken over a closed surface.

The divergence theorem can be also written in coordinate form as

$$\int_S\int P\, dy\, dz + Q\, dx\, dz + R\, dx\, dy = \int\int_G\int \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} + \frac{\partial R}{\partial x}\right) dx\, dy\, dz$$

a near analog of the problem description. In a particular case, by setting P=x, Q=y, R=z, we obtain a formula for the volume of solid G:

$$V = \frac{1}{3}\int_S\int x\, dy\, dz + y\, dx\, dz + z\, dx\, dy$$

and get to the equality posed in the problem. To test in Mathematica terms, there is the command **Div** for divergence,

```
tes = Div[{P[x, y, z], Q[x, y, z], R[x, y, z]}, {x, y, z}]
```

$R^{(0,0,1)}[x, y, z] + Q^{(0,1,0)}[x, y, z] + P^{(1,0,0)}[x, y, z]$

which yields a fair resemblence of the guts of the problem rhs. However, to try to pursue this Mathematica-wise

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \text{tes}\, dx\, dy\, dz$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \left(R^{(0,0,1)}[x, y, z] + Q^{(0,1,0)}[x, y, z] + P^{(1,0,0)}[x, y, z]\right) dx\, dy\, dz$$

lacks quite a bit for going anywhere.

9. Ball. Find the volume under a hemisphere of radius a from the formula in problem 7.

```
Clear["Global`*"]
```

The problem is simplified by the fact that the hemisphere seems centered at the origin. The post by user370967 at *https://math.stackexchange.com/questions/2246169/calculate-the-volume-of-a-sphere-using-only-double-integrals* was useful, especially the part about using polar coordinates. At the present time I can't comply with the problem instructions, using problem 7, so I will just repeat the method from user370967.

The surface of the sphere has the equation

$$x^2 + y^2 + z^2 = a^2$$

which can be rewritten as

$$z = \pm\sqrt{a^2 - x^2 - y^2}$$

By symmetry

$$\text{vol}_{\text{sphere}} = 2\int\int_{gG} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

where G is the implicit region (in ℝ)

```
gG = ImplicitRegion[x^2 + y^2 ≤ a^2, {x, y}];
```

By a change to polar coordinates

```
x = a Cos[θ];
y = a Sin[θ];
```

$$\sqrt{a^2 - \left(x^2 + y^2\right)} = \sqrt{a^2 - r^2}$$

and

$$dx\, dy = r\, d\theta\, dr$$

The volume integral then is

$$\int_0^{2\pi}\int_0^{a} r\sqrt{a^2 - r^2}\, dr\, d\theta$$

$$\frac{2}{3}\left(a^2\right)^{3/2}\pi$$

The above integral result is the expected value for a hemispheric volume.

10. Volume. Show that a region T with boundary surface S has the volume

$$V = \frac{1}{3} \int_S \int r \,\text{Cos}[\phi] \, dA$$

where r is the distance of a variable point P: (x, y, z) on S from the origin O and ϕ is the angle between the directed line OP and the outer normal of S at P.

11. Ball. Find the volume of a ball of radius a from problem 10.

Here is a loopy explanation for the problem solution

```
Clear["Global`*"]
```

First, consider the (legitimate) calculation of the ball's surface

$$\int_0^{2\pi} \int_0^{\pi} a^2 \,\text{Sin}[\theta] \, d\theta \, d\phi$$

$$4 a^2 \pi$$

Now, retaining what S is, its double integral form and the limits, I cram everything in, surface plus proposed volume, and substitute a for r.

$$\frac{1}{3} \int_0^{2\pi} \int_0^{\pi} a \,\text{Cos}[\phi] \, a^2 \,\text{Sin}[\theta] \, d\theta \, d\phi$$

$$0$$

Looking at the limits and differentials, I see they have nothing to do with a, and so I pull the a s out, leaving

$$\frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \text{Cos}[\phi] \,\text{Sin}[\theta] \, d\theta \, d\phi$$

$$0$$

From the text answer, I see that Cos[ϕ] is considered to be 1. I offer this explanation: the problem does not deal solely with spheres; potentially, the surface which is being considered can be any shape. Consider an arbitrary point p_1 on this surface. An outwardly directed normal originates at p_1, call this normal n_1. Now consider the vector, or as the problem terms it, the directed line, from the origin to p_1, call it Op_1. The angle between Op_1 and n_1 is an instance of ϕ, call it ϕ_1. The collection of ϕ_i s actually participates in generating the surface. But in the present case the surface is a sphere, and for every p_i on it, for every ϕ_i of it, $\phi_i = 0$, (since a retro-directed extension of n_1 intersects the origin), thus making $\text{Cos}[\phi_i] = 1$ everywhere. So, proceeding,

$$\frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi}\mathrm{Sin}[\theta]\,d\theta\,d\phi$$

$$\frac{4\,a^3\,\pi}{3}$$

The answer in the green cell matches that of the text.